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Irreducible Components of the Spectrum of an Affine Ring

WILLIAM W. KRAUTER*

Department of Mathematics and Statistics, University of Nebraska, Lincoln, Nebraska 68588

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The topology of spec(R) for R a Noetherian ring is determined by the ordering of spec(R). Thus the question of whether spec(R) is homeomorphic to spec(R') for R, R' Noetherian is equivalent to the question of whether the specs are isomorphic as partially ordered sets. Using partially ordered sets, Wiegand [5] has shown that there is, up to homeomorphism, only one irreducible affine surface over the algebraic closure of a finite field. To classify all affine surfaces over the algebraic closure of a finite field, we must catalog all possible ways in which the components can intersect. The main result of this paper, Theorem 2.3, will catalog how the irreducible components of an affine n-dimensional space over an algebraically closed field can intersect. We apply this result and the result of Wiegand to classify, up to homeomorphism, all affine surfaces over the algebraic closure of a finite field.

1. PRELIMINARIES

A given irreducible component of a surface over the algebraic closure of a finite field might have dimension 0 or 1. It is easy to catalog all possible ways in which these components can intersect. For convenience, we use the word *shrub* for any partially ordered set isomorphic to spec(Z). Thus, our one-dimensional components are all shrubs. We recall from [5] the axioms that characterize the partially ordered set $U = \text{spec}(R)$, where R is a two-dimensional affine domain over the algebraic closure of a finite field:

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(S1) U has a unique minimal element,

(S2) U has dimension two,

(S3) U is countable,

(S4) for each element x of height one, there are infinitely many $y > x$,

(S5) for each pair x, y of distinct elements of height one, there are only finitely many z such that $x < z$ and $y < z$, and

(S6) given a finite set S of height-one elements and a finite set T of maximal elements, there is a height-one element w such that (i) $w < t$ for each $t \in T$, and (ii) if $s \in S$, $s < x$, and $w < x$, then $x \in T$.

It is shown in [5] that any two partially ordered sets satisfying (S1)–(S6) are isomorphic. We will call such a partially ordered set a *bush*. (It is interesting to note that by [6], a bush is simply a partially ordered set isomorphic to spec($Z[X]$).) We now establish some terminoloy and notation to describe how the shrubs and bushes are tied together.

For any partially ordered set U, let $\min(U)$ denote the set of minimal elements of U. If $x, y \in U$, let $\mu(x,y) = \min\{z \in U: z \geqslant x \text{ and } z \geqslant y\}$. If $A \subseteq U$, let $A^{\#} = \bigcup \{\mu(x,y): x, y \in A\}$, and let $A^0 = A \cup A^{\#} \cup A^{\#\#} \cup \cdots$. Thus A^0 is the smallest set containing A and closed under the $\#$ operation.

To classify all affine surfaces over the algebraic closure of a finite field, we will study $(\operatorname{minspec}(A))^0$, where A is a two-dimensional affine ring over the algebraic closure of a finite field. This object will describe how irreducible components of spec(A) fit together. Unfortunately, the partially ordered set structure of $(\operatorname{minspec}(A))^0$ does not distinguish between a surface with two bushes intersecting in a curve, and a surface with two bushes intersecting in a point. We need to know the coheight (in spec(A)) of each member of $(\operatorname{minspec}(A))^0$. We will make these notions precise in Definition 1.1 and 1.2.

DEFINITION 1.1. Let U be a partially ordered set. An *index* on U is a function γ from U to the set of nonnegative integers such that $\gamma(x) < \gamma(y)$ whenever $x > y$. The pair (U, γ) will be called an *indexed partially ordered set*. An *isomorphism* from one indexed partially ordered set to another is an order isomorphism that preserves the index.

DEFINITION 1.2. The *skeleton* of a partially ordered set U is the indexed partially ordered set $((\min(U))^0, \operatorname{coht}_U)$, where $\operatorname{coht}_U(x)$ denotes the coheight in U of the element x. We will drop the U and simply write "coht" where the set is understood to be the same.

Finally, a subset of a partially ordered set X is said to be *closed* provided it is a finite union of sets of the form $\{x\}^- = \{y \in X \mid y \geqslant x\}$. The components of X are the closed sets $\{x\}^-$, where x is a minimal element of

X. In all the partially ordered sets we will encounter, it is easily checked that the closed sets satisfy the axioms for a topology on X (though this certainly is not true in all partially ordered sets). When X is the prime spectrum of a Noetherian ring, we just get the Zariski topology.

Suppose that A is a finitely generated k algebra of dimension two, where k is the algebraic closure of a finite field. The partially ordered set $U = \operatorname{spec}(A)$ satisfies the conditions

(T1) U has only finitely many minimal elements $x_1, \ldots, x_m$,

(T2) each component $\{x_i\}^-$ is either a point, a shrub, or a bush, and

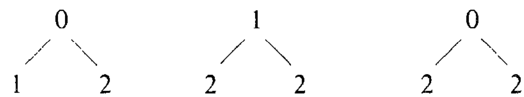
(T3) $\{x_i\}^- \cap \{x_j\}^-$ is a closed subset of $\{x_j\}^-$ for each i, j.

If U is a partially ordered set satisfying (T1)–(T3), then it is easily checked that the closed sets satisfy the axioms for a topology on U. If $A \subseteq U$, we let A^- denote the closure of A with respect to this topology. If $\min(A)$ is finite, then $A^- = \bigcup \{\{x\}^- \mid x \in \min(A)\}$.

There are two steps in classifying, up to homeomorphism, all affine surfaces over the algebraic closure of a finite field. Our first goal is to prove all possible skeletons occur. Later, we will prove that two partially ordered sets satisfying (T1)–(T3) are isomorphic if and only if their skeletons are isomorphic.

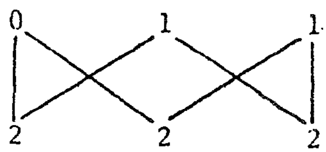
2. The Skeleton Theorem

Consider these skeletons (where the numbers are indices)



$$\frac{k[X, Y, Z]}{(X, Y) \cap (Z)} \qquad \frac{k[X, Y, Z]}{(X) \cap (Y)} \qquad \frac{k[X, Y, Z, W]}{(X, Y) \cap (Z, W)}.$$

These ad hoc constructions, however, do not seem to work for the following example:



Our general construction relies on Bertini's theorem and works in every dimension. In the course of the proof, we will use the expression "for sufficiently general $\alpha_i \in k$." This expression means that the set of such constants $(\alpha_1,..., \alpha_n)$ forms a nonempty open subset of A_k^n. We begin with Lemma 2.1 which is a sharpened form of [5, Lemma 4].

LEMMA 2.1. *Let k be any algebraically closed field, let $A = k[x_1,..., x_n]$ be an affine domain over k and let (f, g) be an A sequence generating a proper ideal of A. Then, for sufficiently general constants $\alpha, \beta_i \in k$ the principal ideal generated by $f + (\alpha + \sum_{i=1}^n \beta_i x_i)g$ is prime.*

Proof. Write $A = k[X_1,..., X_n]/P$, with $X_i \mapsto x_i$, and let F, G be polynomials mapping to f, g, respectively. Let Z be a new indeterminate, and let S be the hypersurface in A_k^{n+1} defined by $F + ZG = 0$. Let V be the variety defined by the extension of P to $k[X_1,..., X_n, Z]$. Since $(f + Zg)$ is a prime ideal in $A[Z]$, by [1, p. 102, Ex. 2], the ideal $P[Z] + (F + ZG)$ is prime, and consequently, the algebraic set $V \cap S$ is a variety where $V \cap S$ is the scheme-theoretic intersection. By [3, Theorem 12], [4, Theorem 1.1], there is a nonempty open set U in A_k^{n+2} such that if $(\alpha', \beta_1',..., \beta_n', \gamma) \in U$, the hyperplane $\gamma Z = \alpha' + \sum_{i=1}^n \beta_i' X_i$ meets $V \cap S$ in a variety. Let $N = \{(a_1,..., a_{n+2}) \in A_k^{n+2} \mid a_{n+2} \neq 0\}$ and let $W = \{(\gamma^{-1}\alpha', \gamma^{-1}\beta_1',..., \gamma^{-1}\beta_n') \mid (\alpha', \beta_1',..., \beta_n', \gamma) \in U \cap N\}$. For each nonzero $\gamma \in k$, let $\phi_\gamma: A_k^{n+1} \to A_k^{n+2}$ be the morphism $(a_1,..., a_{n+1}) \mapsto (\gamma a_1,..., \gamma a_{n+1}, \gamma)$. Clearly, $W = \bigcup_\gamma \phi_\gamma^{-1}(U \cap N)$, and so W is a nonempty open subset of A_k^{n+1}. We will prove that $f + (\alpha + \sum_{i=1}^n \beta_i x_i)g$ generates a prime ideal of A whenever $(\alpha, \beta_1,..., \beta_n) \in W$.

If $(\alpha, \beta_1,..., \beta_n) \in W$, then the hyperplane $Z = \alpha + \sum_{i=1}^n \beta_i X_i$ meets $V \cap S$ in a variety. This means that the ideal $Q = (F + ZG) + P[Z] + (Z - \alpha - \sum_{i=1}^n \beta_i X_i)$ is prime. We observe that $Q \cap k[X_1,..., X_n] = P + (F + (\alpha + \sum_{i=1}^n \beta_i X_i)G)$. One inclusion ($\supseteq$) is easy: $F + (\alpha + \sum_{i=1}^n \beta_i X_i)G = F + ZG - G(Z - \alpha - \sum_{i=1}^n \beta_i X_i) \in Q$. For the reverse inclusion, suppose $H \in Q \cap k[X_1,..., X_n]$, say, $H = (F + ZG)\, h_1 + \sum b_i Z^i + (Z - \alpha - \sum_{i=1}^n \beta_i X_i)\, h_2$, h_1, $h_2 \in k[X_1,..., X_n, Z]$ and $b_i \in P$. If we substitute $Z = \alpha + \sum \beta_i X_i$, then we see that $H \in P + (F + (\alpha + \sum_{i=1}^n \beta_i X_i)G)$ since H does not involve Z.

Now we know that $P + (F + (\alpha + \sum_{i=1}^n \beta_i X_i)G)$ is a prime ideal. By killing P we see that $(f + (\alpha + \sum \beta_i x_i)g)$ is prime in A, as desired.

LEMMA 2.2. *Let k be any algebraically closed field and let $R = k[x_1,..., x_n]$ be a finitely generated k algebra. Let $f, g_1,..., g_m \in R$, and let I be a radical ideal of $k[x_1,..., x_n]$ such that $f \notin I$. Then $f + \alpha_1 g_1 + \cdots + \alpha_m g_m \notin I$ for sufficiently general constants $\alpha_1,..., \alpha_m \in k$.*

Proof. We may assume, without loss of generality, that R is the polynomial ring $k[X_1,..., X_n]$. Let $S \subset A_k^m$ be the set of m-tuples $\alpha = (\alpha_1,..., \alpha_m)$ such that $f + \alpha_1 g_1 + \cdots + \alpha_m g_m \in I$. Let C be the algebraic set in A_k^n defined by I. Let $Y_1,..., Y_m$ be new indeterminates, and write

$h = f + Y_1 g_1 + \cdots + Y_m g_m \in k[X_1,..., X_n, \ Y_1,..., Y_m]$. Then $\alpha \in S$ if and only if $h(X_1,..., X_n, \alpha_1,..., \alpha_m) \in I$. But $h(X_1,..., X_n, \alpha_1,..., \alpha_m) \in I$ if and only if $h(\beta_1,..., \beta_n, \alpha_1,..., \alpha_m) = 0$ for all $(\beta_1,..., \beta_n) \in C$, since I is a radical ideal. Therefore S is the intersection of the zero sets, in A_k^m, of all polynomials $h(\beta_1,..., \beta_n, \ Y_1,..., Y_m)$, as $(\beta_1,..., \beta_n)$ ranges over C; consequently, S is a proper closed subset of A_k^m.

In the next theorem, we will finally show that all conceivable skeletons occur. First, let us recall that if (U, γ) is a skeleton of a two-dimensional affine ring, then (i) U is finite, (ii) $U = (\min(U))^0$, and (iii) $\max\{\gamma(x): x \in U\} = 2$.

THEOREM 2.3. *Let (U, γ) be a finite indexed partially ordered set such that $(\min(U))^0 = U$. Let n be the maximum value of $\gamma(x)$ for $x \in U$. Then, for any algebraically closed field k, there is a $(2n+1)$-generated k algebra whose skeleton is isomorphic to (U, γ).*

Proof. There are two steps: We will find a set V of prime ideals in the polynomial ring $k[X_1,..., X_{2n}]$ such that (U, γ) is isomorphic to $(V, \mathrm{coht}_{k[X_1,...,X_{2n}]})$. Then, we factor out $\cap V$ and localize at a suitable nonzero divisor in order to ensure that $V = (\mathrm{minspec}(R))^0$.

Let $U_i = \gamma^{-1}(i)$, $i = 0, 1, 2,...$. Choose a one-to-one function $u \mapsto \alpha_u$ from $U_0 \cup U_1$ to $k - \{0\}$. We will let each $u \in U_0$ correspond to the maximal ideal $(X_1 - \alpha_u)$ of the ring $k[X_1]$. For each $v \in U_1$, let G_v be the product of the polynomials $X_1 - \alpha_u$, $u \geqslant v$; then $G_v + X_2$ is an irreducible polynomial in the ring $k[X_1, X_2]$. It is trivial to verify that the map defined by $u \mapsto (X_1 - \alpha_u, X_2)$ for $u \in U_0$ and $v \mapsto (G_v + X_2)$ for $v \in U_1$ is an isomorphism from $(U_0 \cup U_1, \gamma)$ onto a subset of the indexed partially ordered set $(\mathrm{spec}(k[X_1, X_2]), \mathrm{coht})$.

Suppose, for some s, $2 \leqslant s \leqslant n$, there is an isomorphism ϕ from $(U_0 \cup \cdots \cup U_{s-1}, \gamma)$ onto a subset V of the indexed partially ordered set $(\mathrm{spec}(k[X_1,..., X_{2s-2}]), \mathrm{coht})$ such that $P \not\subseteq Q$ whenever $P \in V^0 - V$ and $Q \in V$. (We have just produced such an isomorphism for $s = 2$.) For each $u \in U_0 \cup \cdots \cup U_{s-1}$, let $\psi(u)$ be the ideal of $k[X_1,..., X_{2s}]$ generated by $\phi(u)$, X_{2s-1}, and X_{2s}. Clearly, ψ is an isomorphism from $(U_0 \cup \cdots \cup U_{s-1}, \gamma)$ onto a subset V' of the indexed partially ordered set $(\mathrm{spec}(k[X_1,..., X_{2s}]), \mathrm{coht})$. We want to extend the domain of ψ to include the set $U_s = \{w_1,..., w_m\}$.

Assume, inductively, that for some r, $0 \leqslant r \leqslant m$, we have prime ideals Q_i, of the ring $k[X_1,..., X_{2s}]$, $1 \leqslant i \leqslant r$, satisfying the conditions

(A1) each prime Q_i has coheight s,

(A2) the function that assigns u to $\psi(u)$ if $u \in U_0 \cup \cdots \cup U_{s-1}$, and w_i to Q_i defines an isomorphism, say θ, from the finite indexed partially ordered set $(U_0 \cup \cdots \cup U_{s-1} \cup \{w_1,..., w_r\}, \gamma)$ onto a subset W of $(\operatorname{spec}(k[X_1,..., X_{2s}]), \text{coht})$, and

(A3) if $P \in W^0 - W$ and $Q \in W$, then $P \not\subseteq Q$.

For $r = 0$, the conditions are satisfied vacuously. We now will extend the domain of our isomorphism θ to include w_{r+1}.

Let $\mathcal{Cl} = \{u \in U_0 \cup \cdots \cup U_{s-1} \mid u > w_{r+1}\}$ and let $\mathcal{B} = U_0 \cup \cdots \cup U_{s-1} - \mathcal{Cl}$. In the ring $k[X_1,..., X_{2s}]$ pick a nonunit $f_1 \in \bigcap \{\psi(u) \mid u \in \mathcal{Cl}\} - (\bigcup \{\psi(v) \mid v \in \mathcal{B}\} \cup (\bigcup \{Q_i \mid 1 \leqslant i \leqslant r\}))$. If $u \in \mathcal{Cl}$, the prime ideal $\psi(u)$ has coheight $\leqslant s-1$ by condition (A2), so its height is at least $s+1 \geqslant 3$. Therefore, we can certainly find a nonunit $g_1 \in \bigcap \{\Psi(u) \mid u \in \mathcal{Cl}\}$ relatively prime to f_1. That is, (f_1, g_1) is a $k[X_1,..., X_{2s}]$ sequence. Then, by Lemma 2.1, for sufficiently general constants $\alpha_1,..., \alpha_{2s}$, $\beta \in k$, the polynomial $f_1 + (\beta \sum_{i=1}^{2s} \alpha_i X_i)\, g_1 = F_1$ is irreducible. Furthermore, by Lemma 2.2, we may choose our constants in such a way that $F_1 \notin \bigcup \{\psi(v) \mid v \in \mathcal{B}\} \cup \{Q_i \mid 1 \leqslant i \leqslant r\}$.

Assume inductively, that for some j, $1 \leqslant j < s$, there is a sequence of polynomials, $F_1,..., F_j$ in $k[X_1,..., X_{2s}]$ satisfying the two conditions

(B1) for all l, $1 \leqslant l \leqslant j$, the ideal $(F_1,..., F_l)$ is a prime ideal of height l in $k[X_1,..., X_{2s}]$, and $(F_1,..., F_l) \subseteq \psi(u)$ if and only if $w_{r+1} < u$, and

(B2) for each $Q \in W$ and each l, $1 \leqslant l \leqslant j$, each minimal prime ideal of $Q + (F_1,..., F_{l-1})$ that contains the polynomial F_l is in W^0.

For $j = 1$, this has been accomplished. Since each subsequence $(F_1,..., F_l)$ generates a prime ideal, $(F_1,..., F_l)$ is a $k[X_1,..., X_{2s}]$ sequence. Therefore the factor ring $R = k[X_1,..., X_{2s}]/(F_1,..., F_j)$ is Cohen–Macaulay.

For the inductive step, let x_i denote the image of X_i in R; also if I is an ideal of $k[X_1,..., X_{2s}]$, let IR denote the ideal generated by the image of I in R. For each $Q \in W$, let $H(Q)$ be the set of prime ideals of $k[X_1,..., X_{2s}]$ satisfying the following conditions: (i) P is a minimal prime of $Q + (F_1,..., F_j)$; (ii) $P \notin W$; and (iii) for each $u \in \mathcal{Cl}$, $\psi(u) \not\subseteq P$. (Note that $H(Q)$ is a finite set and that PR is a prime ideal of R for each $P \in H(Q)$.)

In the ring $k[x_1,..., x_{2s}]$, pick a nonunit $f_{j+1} \in \bigcap \{\psi(u)R \mid u \in \mathcal{Cl}\} - (\bigcup \{PR \mid P \in H(Q)$ for some $Q \in W\})$. The prime ideal $(F_1,..., F_j)$ has height $j < s$, by condition (B1). For each $u \in U_0 \cup \cdots \cup U_{s-1}$, the prime ideal $\psi(u)$ has coheight $\leqslant s-1$ by condition (A2), so $\psi(u)$ has height at least $s+1$. Since $k[X_1,..., X_{2s}]$ is a catenarian ring, [2, p. 87, (14E)], it follows that for each $u \in \mathcal{Cl}$, the prime ideal $\psi(u)R$ has height at least 2. Since $k[x_1,..., x_{2s}]$ is Cohen–Macaulay, we can pick a nonunit $g_{j+1} \in \bigcap \{\psi(u) \mid u \in \mathcal{Cl}\}$ such that (f_{j+1}, g_{j+1}) is a $k[x_1,..., x_{2s}]$ sequence. Again, by Lemma 2.1, we know that for sufficiently general constants $\alpha_1,..., \alpha_{2s}$, $\beta \in k$, the element $f_{j+1} + (\beta + \sum_{i=1}^{2s} \alpha_i x_i)\, g_{j+1}$ generates a principal prime ideal. Moreover, by Lemma 2.2, we know that for sufficiently general constants, $\alpha_1,\ldots,\alpha_{2s}, \beta\in k$, the element $f_{j+1}+(\beta+\sum_{i=1}^{2s}\alpha_i x_i)\,g_{j+1}$ is not in PR, whenever $P\in H(Q)$ and $Q\in W$. We now fix such constants for the remainder of the proof. Let $F_{j+1}\in k[X_1,\ldots,X_{2s}]$ map onto $f_{j+1}+(\beta+\sum_{i=1}^{2s}\alpha_i x_i)\,g_{j+1}$. The new sequence of polynomials $F_1,\ldots,F_{j+1}$ clearly satisfies condition (B1) of our inductive hypothesis. To show that the sequence $F_1,\ldots,F_{j+1}$ satisfies condition (B2), let $Q\in W$, fix $l\leqslant j+1$, and let P be a minimal prime ideal of $Q+(F_1,\ldots,F_{l-1})$ such that $F_l\in P$. If $l\leqslant j$, then, by the inductive hypothesis, $P\in W^0$. We now assume that $l=j+1$. If $\psi(u)\subseteq P$ for some $u\in \mathcal{U}$, then, since $(F_1,\ldots,F_j)\subseteq\psi(u)$, it follows that P is a minimal prime of the ideal $\psi(u)+Q$. But $\psi(u)$ and Q are in W, so $P\in W^{\#}\subseteq W^0$. Finally, suppose $\psi(u)\nsubseteq P$ for every $u\in\mathcal{U}$. If $P\notin W$, and since P is a minimal prime of $Q+(F_1,\ldots,F_j)$, we see that $P\in H(Q)$. But $F_{j+1}\in P$, and this contradicts our choice of f_{j+1}. This completes the induction step.

We have built a prime ideal $Q_{r+1}=(F_1,\ldots,F_s)$ of coheight s in $k[X_1,\ldots,X_{2s}]$. But also, the function that assigns x to $\psi(x)$ for $x\in U_0\cup\cdots\cup U_{s-1}$, and z_i to Q_i, for $1\leqslant i\leqslant r+1$, is an isomorphism from the finite indexed partially ordered set $(U_0\cup\cdots\cup U_{s-1}\cup\{w_1,\ldots,w_{r+1}\},\gamma')$ onto the subset $W\cup\{Q_{r+1}\}$ of $(\operatorname{spec}(k[X_1,\ldots,X_{2s}]), \operatorname{coht}_{k[X_1,\ldots,X_{2s}]})$. Now we will prove that if $P\in(W\cup\{Q_{r+1}\})^0-(W\cup\{Q_{r+1}\})$ and $Q\in W\cup\{Q_{r+1}\}$, then $P\nsubseteq Q$. First, if $P\in W^0$ and $Q\in W$, then $P\nsubseteq Q$ by our inductive hypothesis. Also, if $P\in W^0$ and $Q=Q_{r+1}$, then the coheight of P is strictly less than the coheight of Q; therefore $P\nsubseteq Q$. The only remaining case is $P\notin W^0$. Suppose, by way of contradiction, that there exists $Q\in W\cup\{Q_{r+1}\}$ such that $P\subseteq Q$. It is clear that $Q\neq Q_{r+1}$, so $Q\in W$. Choose $N_0\subseteq P$ minimal with respect to the property that $N_0\in(W\cup\{Q_{r+1}\})^0-(W^0\cup\{Q_{r+1}\})$. Then $N_0\notin W^0$, and N_0 is a minimal prime of $P'+Q_{r+1}=P'+(F_1,\ldots,F_s)$ for some $P'\in W^0$. Now $P'\subseteq Q\in W$, and condition (A3) of our inductive hypothesis forces P' to be in W.

Assume, inductively, that for some j, $0\leqslant j<s-1$, we have a properly descending chain of prime ideals $Q\supset N_0\supset\cdots\supset N_j$ such that N_j is a minimal prime ideal of $P'+(F_1,\ldots,F_{s-j})$ and $N_j\notin W^0$. For $j=0$ this has been accomplished. Because $N_j\notin W^0$ and $F_{s-j}\in N_j$, we see that N_j cannot be a minimal prime of $P'+(F_1,\ldots,F_{s-j-1})$ by condition (B2). Let N_{j+1} be a minimal prime of $P'+(F_1,\ldots,F_{s-j-1})$ such that $N_{j+1}\subset N_j$. We claim that $N_{j+1}\notin W$. To see this, suppose, by way of contradiction, that $N_{j+1}\in W$. By construction, $F_1\notin Q_i$ for all $i\leqslant r$. Since $N_{j+1}\supseteq P'+(F_1,\ldots,F_{s-j-1})$, we see that $F_1\in N_{j+1}$, and consequently $N_{j+1}\neq Q_i$ for all $i\leqslant r$. Therefore $N_{j+1}=\psi(u)$ for some $u\in U_0\cup\cdots\cup U_{s-1}$: and since $F_1\in N_{j+1}$, condition (B1) of our inductive hypothesis implies that $F_i\in N_{j+1}$ for all $i\leqslant s$. Since N_{j+1} is a minimal prime of $P'+(F_1,\ldots,F_{s-j-1})$, it now follows that $N_{j+1}\supseteq P'+(F_1,\ldots,F_s)$. But this is a contradiction since $N_0\supset N_{j+1}$, and N_0 is a minimal prime of $P'+(F_1,\ldots,F_s)$. Furthermore, since $N_{j+1}\subseteq Q\in W$, we

see that $N_{j+1} \notin W^0 - W$ by condition (A3) of our inductive hypothesis. In other words, $N_{j+1} \notin W^0$, and our induction is complete. We have a properly descending chain of prime ideals $Q \supset N_0 \supset \cdots \supset N_{s-1}$ such that N_{s-1} is a minimal prime of $P' + (F_1)$. But then the coheight of P' is at least $s+1$. Since $P' \in W$ this is a contradiction.

This completes the induction. Taking $s = n$, we get an isomorphism from (U, γ) onto $(W, \mathrm{coht}_{k[X_1,\ldots,X_{2n}]})$ for some subset W of $\mathrm{spec}(k[X_1,\ldots, X_{2n}])$. Moreover, we have $W^0 = (\min(W))^0$, and $P \not\subseteq Q$ whenever $P \in W^0 - W$ and $Q \in W$.

Let $B = k[X_1,\ldots, X_{2n}]/\bigcap W$, and let W' be the canonical image of W in B. Then $\min(W') = \mathrm{minspec}(B)$, and $(W')^0 = (\min(W'))^0 = (\mathrm{minspec}(B))^0$. Furthermore, we still have an isomorphism from (U, γ) onto (W', coht_B). The only problem is that $(\min(W'))^0 = (W')^0$ may be strictly larger than W'. Fortunately, we have arranged things so that $P \not\subseteq Q$ whenever $P \in (W')^0$ and $Q \in W'$. Thus we can choose a nonunit $t \in \bigcap \{P \mid P \in (W')^0 - W'\} - \bigcup \{Q \mid Q \in W'\}$. Then $B[1/t]$ is the homomorphic image of $k[X_1,\ldots, X_{2n}, T]$, and we claim that $B[1/t]$ satisfies all our requirements.

By construction, (U, γ) is isomorphic to $(W, \mathrm{coht}_{k[X_1,\ldots,X_{2n}]})$ and $(W, \mathrm{coht}_{k[X_1,\ldots,X_{2n}]})$ is isomorphic to (W', coht_B). Therefore, to show that (U, γ) is isomorphic to $((\mathrm{minspec}(B[1/t]))^0, \mathrm{coht}_{B[1/t]})$, we will prove that (W', coht_B) is isomorphic to it instead. First we recall that the canonical homomorphism from B to $B[1/t]$ induces an isomorphism, denoted by Γ, from $(\mathrm{spec}(B[1/t]), \mathrm{coht}_{B[1/t]})$ onto $(D(t), \mathrm{coht}_B)$, where $D(t)$ is the set of prime ideals in B not containing t. Moreover, since t is in $\bigcap \{P \mid P \in (W')^0 - W'\} - \bigcup \{Q \mid Q \in W'\}$, we see that $W' \subseteq D(t)$. Therefore we have a canonical embedding from (W', coht_B) into $(D(t), \mathrm{coht}_B)$. Since $(W')^0 = (\min(W'))^0 = (\mathrm{minspec}(B))^0$, it follows that Γ induces an embedding from $((\mathrm{minspec}(B[1/t]))^0, \mathrm{coht}_{B[1/t]})$ into (W', coht_B). Now, we will prove that Γ maps $((\mathrm{minspec}(B[1/t]))^0, \mathrm{coht}_{B[1/t]})$ onto (W', coht_B) by an induction on the heights of the prime ideals of W'. If $Q \in W'$ and the height of Q is 0, then Q is an element of $\mathrm{minspec}(B)$. Since Γ is an isomorphism from $(\mathrm{spec}(B[1/t]), \mathrm{coht}_{B[1/t]})$ onto $(D(t), \mathrm{coht}_B)$, and $W' \subseteq D(t)$ by our choice of t, there is a prime $P \in \mathrm{minspec}(B[1/t])$ such that $\Gamma(P) = Q$. Suppose, inductively, that for some integer $n > 0$, and for every $I \in W'$ of height less than n, there is a prime $J \in (\mathrm{minspec}(B[1/t]))^0$ such that $\Gamma(J) = I$. Let Q be an element of W' of height n. Then, since $W' \subseteq (W')^0 = (\mathrm{minspec}(B))^0$, there exist prime ideals $Q_1, Q_2 \in (W')^0$ such that Q is a minimal prime of $Q_1 + Q_2$ and Q properly contains Q_i, $i = 1, 2$. Since $W' \subseteq D(t)$ and $Q \in W'$, we see that $Q \in D(t)$. Because $Q_i \subset Q$ for $i = 1, 2$, then $Q_i \in D(t)$, $i = 1, 2$. This implies, since t is in $\bigcap \{P \mid P \in (W')^0 - W'\} - \bigcup \{Q \mid Q \in W'\}$ and $Q_i \in (W')^0$, $i = 1, 2$, that $Q_i \in W'$, $i = 1, 2$. By our inductive hypothesis, there are primes P_1 and P_2 in $(\mathrm{minspec}(B[1/t]))^0$ such that $\Gamma(P_i) = Q_i$, $i = 1, 2$. Because Γ is an isomorphism from $\operatorname{spec}(B[1/t])$ onto $D(t)$, there is a prime ideal P of $B[1/t]$ such that $\Gamma(P) = Q$. We claim that $P \in (\operatorname{minspec}(B[1/t]))^0$. To see this, we note that P properly contains P_i, $i = 1, 2$, since Q properly contains Q_i, $i = 1, 2$. Also, since Q is a minimal prime of $Q_1 + Q_2$, it follows that P is a minimal prime of $P_1 + P_2$. But P_1 and P_2 are in $(\operatorname{minspec}(B[1/t]))^0$ by our inductive hypothesis, so $P \in \{P_1, P_2\}^{\#} \subseteq (\operatorname{minspec}(B[1/t]))^0$. This completes the induction step, and we see that Γ maps $((\operatorname{minspec}(B[1/t]))^0, \operatorname{coht}_{B[1/t]})$ onto $(W', \operatorname{coht}_B)$.

Now we will complete our classification of the homeomorphism types of affine surfaces over a finite field. Since Theorem 2.3 tells us that every conceivable skeleton occurs, our goal is to prove that two partially ordered sets satisfying (T1)–(T3) are isomorphic if and only if their skeletons are isomorphic.

3. The Isomorphism Theorem

We begin with Lemma 3.1 which is a sharpened form of [5, Lemma 1].

Lemma 3.1. *Let U and V be bushes, let A_1 and B_1 be closed subsets of U and V, respectively, and let A_2 and B_2 be finite subsets of U and V, respectively. Then any height-preserving isomorphism from $A_1 \cup A_2^0$ onto $B_1 \cup B_2^0$ can be extended to an isomorphism from U onto V.*

Proof. If $A_1 = U$, clearly, $B_1 = V$ and there is nothing to prove. Thus we may assume A_1 and B_1 have dimension $\leqslant 1$. Also, any isolated closed points in A_1 (resp. B_1) might as well be included in A_2 (resp. B_2); so we may assume neither A_1 nor B_1 has any zero-dimensional components. Finally, we may assume, without loss of generality, that A_2 (resp. B_2) does not contain the minimal element. Let θ be an isomorphism from $A = A_1 \cup A_2^0$ onto $B = B_1 \cup B_2^0$. We will first extend θ to an isomorphism θ' from $\bar{A}$ onto $\bar{B}$. Then we will extend θ' to an isomorphism from U onto V. Let S (respec. T) denote the set of height-one elements of $A - A_1$ (resp. $B - B_1$). ("Height" always refers to height in U or V, not relative height.) If x is a height-one element of A, then by axiom (S5), $x \in S$ if and only if $\{x\}^- \cap A$ is finite. Similarly, if y is a height-one element in B, then $y \in T$ if and only if $\{y\}^- \cap B$ is finite. It follows that θ maps S onto T. Since $\min(A)$ and $\min(B)$ are finite sets, $A^- = A \cup S^-$ and $B^- = B \cup T^-$. Therefore, if $S = \varnothing$, then $A^- = A$, $B^- = B$, and θ is an isomorphism from A^- onto B^-.

In general, let n be the number of elements in the set S. Having already done the case $n = 0$, let $n > 0$ and $S = \{x_1, \ldots, x_n\} \subseteq A$. Since θ maps S onto T, then $T = \{\theta(x_1), \ldots, \theta(x_n)\}$. Assume, that for some r, $0 \leqslant r < n$, that we have extended our isomorphism θ to an isomorphism ψ from $D_r = A \cup$

$\{x_1\}^- \cup \cdots \cup \{x_r\}^-$ onto $E_r = B \cup \{\theta(x_1)\}^- \cup \cdots \cup \{\theta(x_r)\}^-$, where $D_0 = A$ and $E_0 = B$. Clearly, ψ is a height-preserving isomorphism. Now consider x_{r+1}. Since $x_{r+1} \in A$ and $\theta(x_{r+1}) \in B$, we can extend our isomorphism ψ to a height-preserving isomorphism ψ' from $D_r \cup \{x_{r+1}\}^-$ onto $E_r \cup \{\theta(x_{r+1})\}^-$ by using an arbitrary bijection between the countably infinite sets $\{x_{r+1}\}^- - D_r$ and $\{\theta(x_{r+1})\}^- - E_r$. This completes the induction step, and we now have height-preserving isomorphism θ' from D_n onto E_n. But since $A^- = A \cup S^-$ and $B^- = B \cup T^-$, we see that $A^- = D_n$ and $B^- = E_n$. Therefore θ' is an isomorphism from A^- onto B^- extending θ.

Now we will extend our isomorphism θ' from A^- onto B^- to an isomorphism from U onto V. Changing notation, we let $A = A^-$ and $B = B^-$. Assume, first, that each irreducible component of A is a shrub. Let S and T denote the sets of height-one elements of U and V, respectively. Endow the sets S and T with well orderings of order-type ω. Let S_0 and T_0 denote the sets of minimal elements of A and B, respectively. Then $S_0^- = A$, $T_0^- = B$, and θ' is an isomorphism from S_0^- onto T_0^-. Let x be the first element of $S - S_0$. Then, $\{x\}^- \cap S_0^-$ is finite by axiom (S5). By axiom (S6), there is an $x' \in T$ such that $\theta'(\{x\}^- \cap S_0^-) = \{x'\}^- \cap T_0^-$. We can extend θ' to an isomorphism ψ from $S_0^- \cup \{x\}^-$ onto $T_0^- \cup \{x'\}^-$ by sending x to x', and by using an arbitrary bijection between $\{x\}^- - S_0^-$ and $\{x'\}^- - T_0^-$. Let $S_1 = S_0 \cup \{x\}$, $T_1 = T_0 \cup \{x'\}$, and let y' be the first element of $T - T_1$. By the same process, we can extend ψ^{-1} to an isomorphism from $T_1^- \cup \{y'\}$ onto $S_1^- \cup \{y\}^-$, where y is a suitable member of $S - S_1$. After a countable number of steps of this kind, we get a bijection θ'' from U onto V, where θ'' sends the minimal element of U to the minimal element of V and induces a bijection from S onto T. Since, for each $n \geqslant 0$, θ'' restricted to S_n^- and T_n^- is an isomorphism, it follows that θ'' is an isomorphism from U onto V.

In the general case, let n be the number of zero-dimensional components of A; we will proceed by induction on n. Having already done the case $n = 0$, we assume that $n > 0$, and that any height-preserving isomorphism between two closed subsets D and E of U and V, respectively, can be extended to an isomorphism from U onto V, whenever the number of zero-dimensional components of D is less than n. Since $n > 0$, let $\{x\}$ be a zero-dimensional component of A. By axiom (S6), there are elements $w \in S - S_0$, $w' \in T - T_0$, such that $A \cap \{w\}^- = \{x\}$, and $B \cap \{w'\}^- = \{\theta'(x)\}$. We can extend our isomorphism θ' to a height-preserving isomorphism ψ from $A \cup \{w\}^-$ onto $B \cup \{w'\}^-$ by using an arbitrary bijection between $\{w\}^- - \{x\}$ and $\{w'\}^- - \{\theta'(x)\}$. Now we can apply our inductive hypothesis to $A \cup \{w\}^-$ to extend our isomorphism ψ from $A \cup \{w\}^-$ onto $B \cup \{w'\}^-$ to an isomorphism θ'' from U onto V.

We remark that Lemma 3.1 gives another proof that all bushes are isomorphic.

THEOREM 3.2. *Let U and V be partially ordered sets satisfying* (T1)–(T3). *Then U and V are isomorphic if and only if their skeletons are isomorphic.*

Proof. If U and V are isomorphic, then clearly their skeletons are isomorphic.

Conversely, let A and B be the skeletons of U and V, respectively, and let θ be an isomorphism from A onto B. Let $x_1,..., x_n$ be the minimal elements of U, let $\theta(x_i) = y_i$, $X_i = \{x_i\}^-$, and $Y_i = \{y_i\}^-$. Since θ preserves coheight, we know that $\dim X_i = \dim Y_i$ for all $i = 1, 2,..., n$. We will show that for each r, $0 \leqslant r \leqslant n$, there is an isomorphism ϕ_r from $X_1 \cup \cdots \cup X_r$ onto $Y_1 \cup \cdots \cup Y_r$ agreeing with θ on $A \cap (X_1 \cup \cdots \cup X_r)$. Since $U = X_1 \cup \cdots \cup X_n$ and $V = Y_1 \cup \cdots \cup Y_n$ this will complete the proof.

If $r = 0$, there is nothing to prove. Assume, inductively, that for some r, $0 \leqslant r \leqslant n$, Γ is an isomorphism from $X_1 \cup \cdots \cup X_r$ onto $Y_1 \cup \cdots \cup Y_r$ such that Γ agrees with θ on $A \cap (X_1 \cup \cdots \cup X_r)$. We claim that Γ carries the closed set $(X_1 \cup \cdots \cup X_r) \cap X_{r+1}$ onto the closed set $(Y_1 \cup \cdots \cup Y_r) \cap Y_{r+1}$. To verify this, let $x \in (X_1 \cup \cdots \cup X_r) \cap X_{r+1}$, say $x \in X_i \cap X_{r+1}$. Then $x \geqslant x'$ for some $x' \in \mu(x_i, x_{r+1})$. But then $x' \in A$, and we have $\Gamma(x) \geqslant \Gamma(x') \geqslant \theta(x_{r+1}) = y_{r+1}$; that is, $\Gamma(x) \in (Y_1 \cup \cdots \cup Y_r) \cap Y_{r+1}$. The same argument applied to Γ^{-1} completes the proof of our claim.

Let Γ': $(X_1 \cup \cdots X_r) \cap X_{r+1} \to (Y_1 \cup \cdots \cup Y_r) \cap Y_{r+1}$ be the isomorphism induced by Γ. We also have an isomorphism θ': $A \cap X_{r+1} \to B \cap Y_{r+1}$ induced by θ. Moreover, Γ' and θ' agree on the intersection of their domains, so we have an isomorphism $\Gamma' \cup \theta'$ from $S = ((X_1 \cup \cdots \cup X_r) \cap X_{r+1}) \cup (A \cap X_{r+1})$ onto $T = ((Y_1 \cup \cdots \cup Y_r) \cap Y_{r+1}) \cup (B \cap Y_{r+1})$. If, now, X_{r+1} has dimension 0 or 1, it is a simple matter to extend $\Gamma' \cup \theta'$ to an isomorphism ψ from X_{r+1} onto Y_{r+1}. Then $\Gamma \cup \psi$ is an isomorphism from $X_1 \cup \cdots \cup X_{r+1}$ onto $Y_1 \cup \cdots \cup Y_{r+1}$ agreeing with θ on $A \cap (X_1 \cup \cdots \cup X_{r+1})$. If, instead, X_{r+1} has dimension two, we will apply Lemma 3.1 to get an extension ψ: $X_{r+1} \to Y_{r+1}$. Since $(X_1 \cup \cdots \cup X_r) \cap X_{r+1}$ and $(Y_1 \cup \cdots \cup Y_r) \cap Y_{r+1}$ are closed subsets of X_{r+1} and Y_{r+1}, respectively, it suffices to prove that $(A \cap X_{r+1})^{\#} = (A \cap X_{r+1})$ and $(B \cap Y_{r+1})^{\#} = B \cap Y_{r+1}$ as subsets of X_{r+1} and Y_{r+1}, respectively.

Let $x, y \in A \cap X_{r+1}$, and $z \in \mu(x, y)$. Then $z \in X_{r+1}$ since X_{r+1} is closed. Moreover, since $A^{\#} = A$ and $x, y \in A$, then $z \in A$ as well. Thus $(A \cap X_{r+1})^{\#} = A \cap X_{r+1}$. The same argument shows that $(B \cap Y_{r+1})^{\#} = B \cap Y_{r+1}$, and the proof is complete.

COROLLARY 3.3. *For $i = 1, 2$, let k_i be a finite field or the algebraic closure of a finite field, and let R_i be a finitely generated k_i algebra of dimension two. Then* $\operatorname{spec}(R_1)$ *and* $\operatorname{spec}(R_2)$ *are isomorphic as partially* *ordered sets if and only if their skeletons are isomorphic as indexed partially ordered sets.*

Proof. If P_i is a prime of coheight 2 in R_i, then spec(R_i/P_i) is a bush, by [5, Theorem 5]. Therefore spec(R_1) and spec(R_2) both satisfy (T1)–(T3). By Theorem 3.2, spec(R_1) is isomorphic to spec(R_2).

We have completed our classification of the homeomorphism types of affine surfaces over a finite field. In particular, Theorem 2.3 tells us that every conceivable two-dimensional skeleton occurs as the skeleton of a k algebra generated by five elements, and Theorem 3.2 tells us that each surface is determined by its skeleton.

Theorem 2.3 poses a fundamental question. Given our finite indexed partially ordered set (U, γ) in Theorem 2.3, can we find a radical ideal I in the ring $k[X_1,..., X_{2n}]$ such that $((\text{minspec}(k[X_1,..., X_{2n}]/I))^0, \text{coht}) \cong (U, \gamma)$? The question is particularly interesting when $n = 1$, since two curves (over any field) are homeomorphic if and only if their skeletons are isomorphic. So we are asking whether every affine curve is homeomorphic to a plane curve. When k is the algebraic closure of a finite field, the answer is "yes" by [5]. Also, when k is the algebraic closure of a finite field, the question for $n = 2$ asks whether every affine surface can be embedded, up to homeomorphism, in A_k^4 since by Corollary 3.3, the homeomorphism type of a surface is determined by its skeleton.

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